ON THE WARING–GOLDBACH PROBLEM FOR SEVENTH AND HIGHER POWERS

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Abstract. We apply recent progress on Vinogradov's mean value theorem to improve bounds for the function $H(k)$ in the Waring–Goldbach problem. We obtain new results for all exponents $k \geq 7$, and in particular establish that for large k one has

$$H(k) \leq (4k-2)\log k - (2\log 2 - 1)k - 3.$$

1. Introduction

In our recent work [7], we reported on the consequences for the Waring-Goldbach problem of recent progress on Vinogradov's mean value theorem based on efficient congruencing (see, for example, [9, 10]). We now revisit our analysis in order to incorporate the latest developments stemming from work of Bourgain, Demeter and Guth [1]. We first recall the definition of the function $H(k)$ associated with the Waring-Goldbach problem. Consider a natural number k and prime number p, and define $\theta = \theta(k,p)$ to be the integer with $p^\theta | k$ but $p^{\theta+1} \nmid k$, and $\gamma = \gamma(k,p)$ by

$$\gamma(k,p) = \begin{cases} \theta + 2, & \text{when } p = 2 \text{ and } \theta > 0, \\ \theta + 1, & \text{otherwise.} \end{cases}$$

We then put $K(k) = \prod_{(p-1)|k} p^\gamma$, and denote by $H(k)$ the least integer s such that every sufficiently large positive integer congruent to s modulo $K(k)$ may be written as the sum of s k-th powers of prime numbers.

Improving on the bound $H(k) \leq k(4\log k + 2\log\log k + O(1))$, as $k \to \infty$, due to Hua [3, 4], we recently showed that $H(k) \leq (4k-2)\log k + k - 7$. The improved bound that we now present in this note saves roughly $(2\log 2)k$ further variables.

Theorem 1. *When k is large, one has $H(k) \leq (4k-2)\log k - (2\log 2 - 1)k - 3$.*

For small values of k one has the bounds

$$H(1) \leq 3, \quad H(2) \leq 5, \quad H(3) \leq 9, \quad H(4) \leq 13, \quad H(5) \leq 21, \quad H(6) \leq 32, \quad H(7) \leq 46,$$

as a consequence of work of Vinogradov [8], Hua [2], Kawada and the second author [5], the first author [6], and Zhao [11]. For larger values of k, we recently established that

$$H(8) \leq 61, \quad H(9) \leq 75, \quad H(10) \leq 89, \quad H(11) \leq 103, \quad H(12) \leq 117,$$
$$H(13) \leq 131, \quad H(14) \leq 147, \quad H(15) \leq 163, \quad H(16) \leq 178,$$
$$H(17) \leq 194, \quad H(18) \leq 211, \quad H(19) \leq 227, \quad H(20) \leq 244.$$

We now obtain the following bounds for $H(k)$ when $7 \leq k \leq 20$.

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Theorem 2. *Let $7 \le k \le 20$. Then $H(k) \le s(k)$, where $s(k)$ is defined by Table 1.*

k	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$s(k)$	45	57	69	81	93	107	121	134	149	163	177	193	207	223

Table 1. Upper bounds for $H(k)$ when $7 \le k \le 20$

Our proof of Theorems 1 and 2 proceeds by directly incorporating the refinements available via [1] into our previous methods from [7]. We record improved estimates for Weyl sums in §2, both pointwise bounds and mean value estimates. Then, in §3, we indicate how to refine our previous bounds for $H(k)$ using these bounds, thereby establishing Theorems 1 and 2.

Throughout this paper, the letter ε denotes a sufficiently small positive number. Whenever ε occurs in a statement, we assert that the statement holds for each positive ε, and any implied constant in such a statement is allowed to depend on ε. The letter p, with or without subscripts, is reserved for prime numbers. We also write $e(x)$ for $\exp(2\pi i x)$, and (a, b) for the greatest common divisor of a and b. Finally, for real numbers θ, we denote by $\lfloor \theta \rfloor$ the largest integer not exceeding θ, and by $\lceil \theta \rceil$ the least integer no smaller than θ.

2. Auxiliary estimates for exponential sums

We refine the work of [7, §§2 and 3] by incorporating recent progress on Vinogradov's mean value theorem due to Bourgain, Demeter and Guth [1]. Recall the classical Weyl sum

$$f_k(\alpha; X) = \sum_{X < x \le 2X} e\left(\alpha x^k\right),$$

in which we suppose that $k \ge 2$ is an integer and α is real. When $k \ge 3$ is an integer, we define σ_k by means of the relation

$$\sigma_k^{-1} = \min\left\{2^{k-1}, k(k-1)\right\}. \tag{2.1}$$

Also, for $k \ge 3$, we define the multiplicative function $w_k(q)$ by taking

$$w_k(p^{uk+v}) = \begin{cases} kp^{-u-1/2}, & \text{when } u \ge 0 \text{ and } v = 1, \\ p^{-u-1}, & \text{when } u \ge 0 \text{ and } 2 \le v \le k. \end{cases}$$

Lemma 2.1. *Suppose that $k \ge 3$. Then either one has $f_k(\alpha; X) \ll X^{1-\sigma_k+\varepsilon}$, or there exist integers a and q such that $1 \le q \le X^{k\sigma_k}$, $(a, q) = 1$ and $|q\alpha - a| \le X^{-k+k\sigma_k}$, in which case*

$$f_k(\alpha; X) \ll \frac{w_k(q)X}{1 + X^k|\alpha - a/q|} + X^{1/2+\varepsilon}.$$

Proof. One may apply the argument of the proof of [7, Lemma 2.1], noting only that the refinement of [10, Theorem 11.1] that follows by employing the bounds recorded in [1, Theorem 1.1] permits the use of the exponent σ_k with the revised definition (2.1) presented above. □

We also require upper bounds for the corresponding Weyl sum over prime numbers,

$$g_k(\alpha; X) = \sum_{X < p \le 2X} e\left(\alpha p^k\right),$$

and these we summarise in the next lemma.

Lemma 2.2. *Suppose that $k \geq 4$ and $X^{2\sigma_k/3} \leq P \leq X^{9/20}$. Then either one has the bound $g_k(\alpha; X) \ll X^{1-\sigma_k/3+\varepsilon}$, or else there exist integers a and q such that $1 \leq q \leq P$, $(a, q) = 1$ and $|q\alpha - a| \leq PX^{-k}$, in which case*

$$g_k(\alpha; X) \ll \frac{X^{1+\varepsilon}}{(q + X^k|q\alpha - a|)^{1/2}}. \tag{2.2}$$

Proof. One may follow the argument of the proof of [7, Lemma 2.2], noting that the refinement to the exponent σ_k made available via (2.1) as exhibited in Lemma 2.1. □

In order to describe our critical mean-value estimate, we introduce a set of admissible exponents for kth powers as follows. Let $t = t_k$ and $u = u_k$ be positive integers to be fixed in due course. Put $\theta = 1 - 1/k$, and define

$$\lambda_i = (\theta + \sigma_{k-1}/k)^{i-1} \quad (1 \leq i \leq u + 1). \tag{2.3}$$

Then define $\lambda_{u+2}, \dots, \lambda_{u+t}$ by putting

$$\lambda_{u+2} = \frac{k^2 - \theta^{t-3}}{k^2 + k - k\theta^{t-3}} \lambda_{u+1}, \tag{2.4}$$

$$\lambda_{u+j} = \frac{k^2 - k - 1}{k^2 + k - k\theta^{t-3}} \theta^{j-3} \lambda_{u+1} \quad (3 \leq j \leq t), \tag{2.5}$$

and then write

$$\Lambda = \lambda_1 + \ldots + \lambda_{t+u}. \tag{2.6}$$

Lemma 2.3. *Let k, t and u be positive integers with $k \geq 3$ and $t \geq \left\lfloor \frac{1}{2}(k+3) \right\rfloor$, and let w be a non-negative integer. Define the exponents λ_j and Λ by means of* (2.3)-(2.6), *and put $\eta = \max\{0, k - \Lambda - 2w\sigma_k\}$. Then when N is sufficiently large, one has*

$$\int_0^1 |g_k(\alpha; N)|^{2w} \prod_{j=1}^{t+u} \left| g_k(\alpha; N^{\lambda_j}) \right|^2 \mathrm{d}\alpha \ll N^{2\Lambda + 2w - k + \eta + \varepsilon}.$$

Proof. This is [7, Lemma 3.3], modified to reflect the improved Weyl exponent (2.1) as exhibited in Lemmata 2.1 and 2.2. □

3. The upper bound for $H(k)$

An upper bound for $H(k)$ follows by combining the mean value estimate supplied by Lemma 2.3 with the Weyl-type estimate stemming from Lemma 2.2.

Lemma 3.1. *Let k, t and u be positive integers with $k \geq 3$ and $t \geq \left\lfloor \frac{1}{2}(k+3) \right\rfloor$. Define the exponent Λ by means of* (2.6), *and put $v = \lfloor (k - \Lambda)/(2\sigma_k) \rfloor$ and $\eta^* = k - \Lambda - 2v\sigma_k$. Finally, define*

$$h = \begin{cases} 1, & \text{when } 0 \leq \eta^* < \frac{1}{2}\sigma_k, \\ 2, & \text{when } \frac{1}{2}\sigma_k \leq \eta^* < \sigma_k, \\ 3, & \text{when } \sigma_k \leq \eta^* < 2\sigma_k. \end{cases}$$

Suppose in addition that $2(t + u + v) + h \geq 3k + 1$ and, when $h \in \{1, 2\}$, that either $v \geq 3$ or $\eta^* < h\sigma_k/3$. Then

$$H(k) \leq 2(t + u + v) + h.$$

Proof. This is [7, Lemma 4.1], modified to reflect the improved Weyl exponent (2.1) as exhibited in Lemma 2.1-2.3. $\square$

We establish Theorems 1 and 2 by applying Lemma 3.1. Recall (2.3)-(2.6), and write $\sigma = \sigma_{k-1}$ and $\phi = \theta + \sigma/k$. Then, just as in the discussion of [7, §5], one has

$$k - \Lambda = -\frac{k\sigma}{1-\sigma} + \left(\frac{k^2(k+1)\sigma + \theta^{t-3}((k^3 - 3k^2 + k + 2) - \sigma(k^3 - 2k^2 + k + 2))}{(k^2 + k - k\theta^{t-3})(1-\sigma)} \right) \phi^u.$$

The proof of Theorem 2. Let k be an integer with $7 \le k \le 20$, and define $t = t_k$, $u = u_k$, $v = v_k$ and $h = h_k$ by means of Table 2. Then by application of a simple computer program, one confirms the validity of the hypotheses of Lemma 3.1. Thus $H(k) \le 2(t + u + v) + h$. Indeed, with h_k^* defined as in Table 3, one finds that for each k one has $2\eta^*/\sigma_k < h_k^*$. We note in this context that the entries in this table have been rounded up in the final decimal place presented. This completes our proof of Theorem 2.

k	7	8	9	10	11	12	13	14	15	16	17	18	19	20
t_k	7	12	17	10	13	10	24	19	30	17	25	18	29	37
u_k	13	12	14	25	29	37	28	42	41	60	56	74	66	63
v_k	2	4	3	5	4	6	8	5	3	4	7	4	8	11
h_k	1	1	1	1	1	1	1	2	1	1	1	1	1	1

TABLE 2. The values of t_k, u_k, v_k and h_k for $7 \le k \le 20$

k	7	8	9	10	11	12	13
h_k^*	0.44643	0.22927	0.02678	0.00739	0.97975	0.00042	0.08628
k	14	15	16	17	18	19	20
h_k^*	1.94435	0.03925	0.01091	0.39085	0.00541	0.52855	0.00043

TABLE 3. The values of h_k^* for $7 \le k \le 20$

$\square$

As we remarked in [7, §5], the non-monotonicity in the values of t_k, u_k and v_k recorded in Table 2 is a consequence of the fact that θ and ϕ are close in size, and thus the optimisation is sensitive only to the sum $t_k + u_k$ rather than the individual values of t_k and u_k.

The proof of Theorem 1. We adapt the proof of [7, Theorem 1], supposing throughout that k is sufficiently large. Put $t = t_k$ and $u = u_k$, where

$$t_k = \left\lceil \tfrac{1}{2} k \log k \right\rceil \quad \text{and} \quad u_k = \lceil k(2\log k - \log 2) \rceil - t.$$

It is convenient to define $\gamma = \lceil k(2\log k - \log 2)\rceil - k(2\log k - \log 2)$. Also, we write

$$\tau = \frac{1}{k(k-1)} \quad \text{and} \quad \sigma = \frac{1}{(k-1)(k-2)},$$

so that $\sigma_k = \tau$ and $\sigma_{k-1} = \sigma$. Our earlier formula for $k - \Lambda$ now takes the shape

$$k - \Lambda = -\frac{k\sigma}{1-\sigma} + \left(\frac{k^2(k+1)(k-1)^3\sigma + \theta^t k^3(k^3 - 3k^2 + O(k))}{(k-1)^3(k^2 + k - k\theta^{t-3})(1-\sigma)} \right) \phi^u.$$

As in the corresponding proof of [7, Theorem 1], one finds that

$$\theta^t = e^{-t/k} \left(1 - \frac{\log k}{4k} + O(k^{-3/2}) \right) \asymp k^{-1/2}.$$

Also, since

$$\log \phi = \log \left(1 - \frac{1-\sigma}{k} \right) = -\frac{1}{k} - \frac{1}{2k^2} + O\left(\frac{1}{k^3} \right),$$

one discerns that

$$\phi^u = e^{-u/k} \left(1 - \frac{3\log k - 2\log 2}{4k} + O(k^{-3/2}) \right) \ll k^{-3/2}.$$

Consequently,

$$k - \Lambda = -\frac{k\sigma}{1-\sigma} + \left(k - 1 + O(k^{-1/2}) \right) \theta^t \phi^u + O(k^{-5/2}),$$

where

$$\begin{aligned} \theta^t \phi^u &= e^{-(t+u)/k} \left(1 - \frac{2\log k - \log 2}{2k} + O(k^{-3/2}) \right) \\ &= e^{-\gamma/k} \left(\frac{2}{k^2} - \frac{2\log k - \log 2}{k^3} + O(k^{-7/2}) \right). \end{aligned}$$

Since

$$\frac{\sigma}{\tau} = \frac{k(k-1)}{(k-1)(k-2)} = 1 + \frac{2}{k} + O\left(\frac{1}{k^2} \right),$$

we find that

$$\begin{aligned} \frac{k-\Lambda}{2\tau} &= -\tfrac{1}{2}(k+2) + \frac{e^{-\gamma/k} k(k-1)}{k^3} (k-1)(k - \log k + \tfrac{1}{2}\log 2) + O(k^{-1/2}) \\ &= -\tfrac{1}{2}(k+2) + (k - \log k + \tfrac{1}{2}\log 2 - 2)(1 - \gamma/k) + O(k^{-1/2}) \\ &= \tfrac{1}{2}k - \log k - 3 + \tfrac{1}{2}\log 2 - \gamma + O(k^{-1/2}). \end{aligned}$$

Put $v = \lfloor (k-\Lambda)/(2\tau) \rfloor$, set $\eta^* = k - \Lambda - 2v\tau$, and define h as in the statement of Lemma 3.1. Then one has $0 \le \eta^* < 2\tau$, and in all circumstances one may confirm that

$$2v + h = \frac{k - \Lambda - \eta^*}{\tau} + h \le \frac{k-\Lambda}{\tau} + 2 \le k - 2\log k - 4 + \log 2 - 2\gamma + O(k^{-1/2}).$$

Since

$$2(t+u+v) + h \le 2(2k\log k - k\log 2 + \gamma) + k - 2\log k - 4 + \log 2 - 2\gamma + O(k^{-1/2}),$$

we therefore conclude from Lemma 3.1 that

$$H(k) \le (4k-2)\log k - (2\log 2 - 1)k - 4 + \log 2 + O(k^{-1/2}).$$

We have assumed k to be sufficiently large, and thus we have established the bound

$$H(k) \le (4k-2)\log k - (2\log 2 - 1)k - 3.$$

This completes the proof of Theorem 1. □

References

[1] J. Bourgain, C. Demeter and L. Guth, *Proof of the main conjecture in Vinogradov's mean value theorem for degrees higher than three*, preprint available as arXiv:1512.01565.

[2] L. K. Hua, *Some results in prime number theory*, Quart. J. Math. Oxford Ser. **9** (1938), 68–80.

[3] L. K. Hua, *Additive Primzahltheorie*, B. G. Teubner Verlagsgesellschaft, Leipzig, 1959.

[4] L. K. Hua, *Additive Theory of Prime Numbers*, American Mathematical Society, Providence, RI, 1965.

[5] K. Kawada and T. D. Wooley, *On the Waring–Goldbach problem for fourth and fifth powers*, Proc. London Math. Soc. (3) **83** (2001), no. 1, 1–50.

[6] A. Kumchev, *The Waring–Goldbach problem for seventh powers*, Proc. Amer. Math. Soc. **133** (2005), no. 10, 2927-2937.

[7] A. V. Kumchev and T. D. Wooley, *On the Waring-Goldbach problem for eighth and higher powers*, J. London Math. Soc. (to appear), preprint available as arXiv:1510.00982.

[8] I. M. Vinogradov, *Representation of an odd number as the sum of three primes*, Dokl. Akad. Nauk SSSR **15** (1937), 291–294.

[9] T. D. Wooley, *Vinogradov's mean value theorem via efficient congruencing*, Ann. of Math. (2) **175** (2012), no. 3, 1575–1627.

[10] T. D. Wooley, *Multigrade efficient congruencing and Vinogradov's mean value theorem*, Proc. London Math. Soc. (3) **111** (2015), no. 3, 519–560.

[11] L. Zhao, *On the Waring–Goldbach problem for fourth and sixth powers*, Proc. London Math. Soc. (3) **108** (2014), no. 6, 1593–1622.

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